Algebraic Footprints of Quantum Gravity: a Stability Point of View

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Reference: *Int. J. Mod. Phys. D* **13/10** (2004) 2003–2034 (`hep-th/0410212`)

Contents

Motivation

- Non-commutative spacetime

$$[X_\mu,\, X_\nu] = ?$$

- Modified dispersion relations

$$E^2 = \mathbf{p}^2 + m^2 + ?$$

- Preferred frames — Lorentz symmetry violation
- Invariant length scale

$$\text{Quantum Gravity} \quad \longrightarrow \quad \ell_{\mathrm{P}} \equiv \sqrt{\frac{\hbar G}{c^3}}$$

$$+$$

Lorentz contraction

$$\Downarrow$$

?

The stability criterion

Galileo		Einstein
$[J_a, J_b] = i\,\epsilon_{ab}{}^c J_c$		$[J_a, J_b] = i\,\epsilon_{ab}{}^c J_c$
$[J_a, K_b] = i\,\epsilon_{ab}{}^c K_c$	$\xrightarrow{\text{stabilize}}$	$[J_a, K_b] = i\,\epsilon_{ab}{}^c K_c$
$[K_a, K_b] = 0$		$[K_a, K_b] = i\,t\,\epsilon_{ab}{}^c J_c$

Newton		Heisenberg
$[f(q,p), g(q,p)] = 0$	$\xrightarrow{\text{stabilize}}$	$[f(q,p), g(q,p)] = i\,\hbar\,\{f(q,p), g(q,p)\}$

Lie Algebra Deformations

- **Lie algebras**

Lie algebra (V, μ)

V: finite-dimensional vector space (over $\mathbb{R}$)

μ: *Lie product* $\mu\colon V \times V \to V$

bilinear: $\mu(\lambda x + \rho y) = \lambda\mu(x) + \rho\mu(y)$

antisymmetric: $\mu(x, y) = -\mu(y, x)$

Jacobi: $\mu(x, \mu(y, z)) = \mu(\mu(x, y), z) + \mu(y, \mu(x, z))$

Basis $\{T_A\}$, $A = 1, \ldots, n$ of V $\quad\Rightarrow\quad$ *structure constants* $f_{AB}{}^C$ s.t.

$$[T_A, T_B] \equiv i\,\mu(T_A, T_B) = i\, f_{AB}{}^C T_C$$

Jacobi: $\boxed{f_{AR}{}^S f_{BC}{}^R + f_{BR}{}^S f_{CA}{}^R + f_{CR}{}^S f_{AB}{}^R = 0}$ (relax)

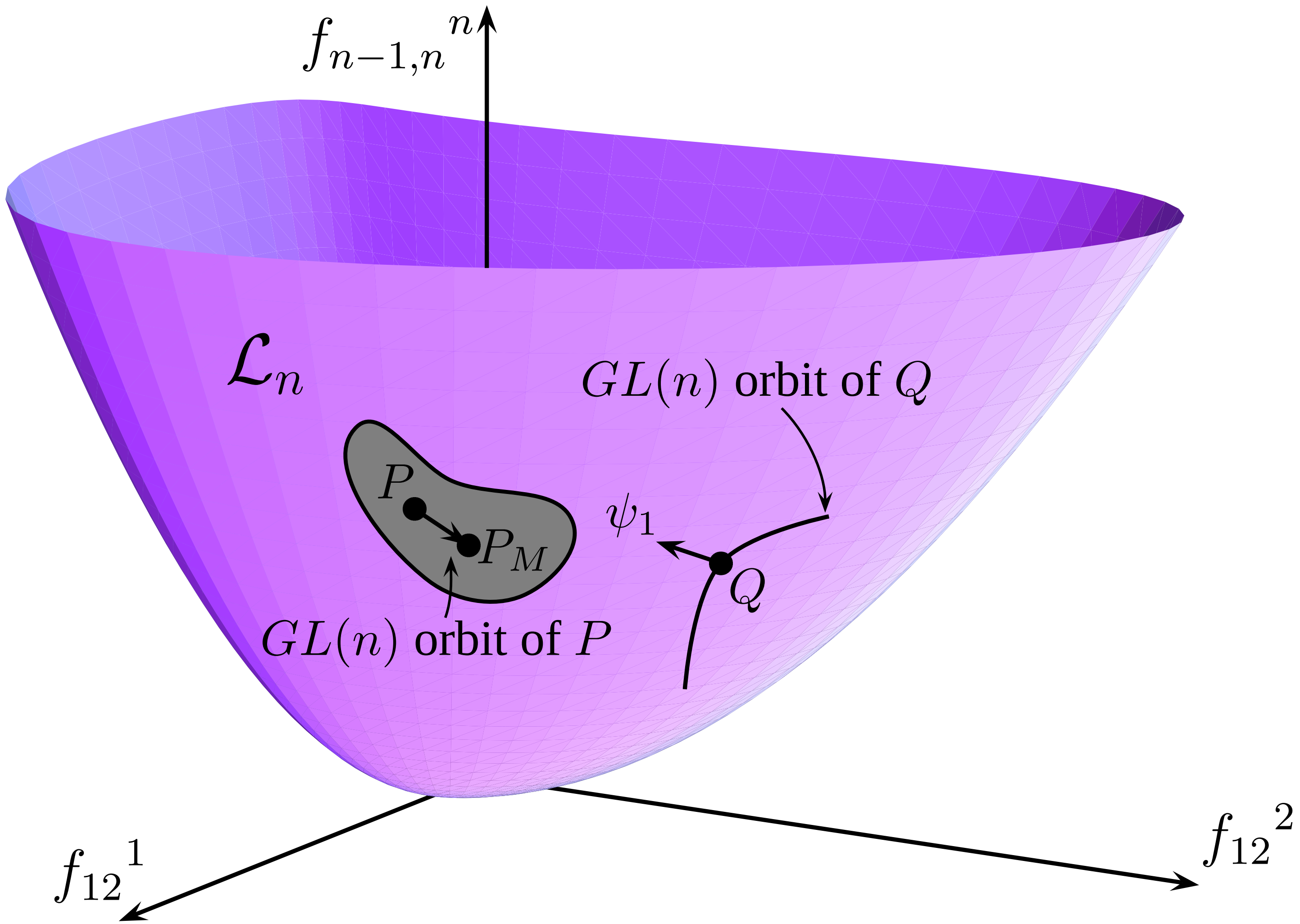

Figure 1: The space $\mathcal{L}_n$ of n-dimensional Lie algebras (sketch).

$$T'_A = M_A{}^B T_B \qquad \Rightarrow \qquad f'_{AB}{}^C = M_A{}^R M_B{}^S (M^{-1})_U{}^C f_{RS}{}^U$$

Orb(P) open $\Rightarrow$ $\mathcal{G}_P$ *stable* (*rigid*), otherwise *unstable*

• Deformations

$\mathcal{G}_0 = (V, \mu_0)\,, \qquad \mu_0(X, Y) \equiv [X,\, Y]_0$

One-parameter (formal) deformation of $\mathcal{G}_0$:

deformed commutator : $$[X,\, Y]_t = [X,\, Y]_0 + \sum_{m=1}^{\infty} \psi_m(X,\, Y)\, t^m$$

t-dependent f's : $$[T_A,\, T_B]_t = i\, f^t_{AB}{}^C T_C$$

$\psi_m : V \times V \to V$, bilinear, antisymmetric (*2-cochains over* V)

Vector space of *p-cochains*: $C^p(V)$

1-cochains: $V \to V$ linear, $C^1(V) \sim \mathrm{Aut}(V)$

0-cochains: constant maps, $C^0(V) \sim V$

- ## Coboundary operator

For any μ, *coboundary operator* $s_\mu : C^p \longrightarrow C^{p+1}$,

$$
\begin{aligned}
s_\mu \triangleright \psi^{(p)}(T_{A_0}, \ldots, T_{A_p}) &= \sum_{r=0}^{p} (-1)^r \mu\left(T_{A_r},\, \psi^{(p)}(T_{A_1}, \ldots, \hat{T}_{A_r}, \ldots, T_{A_p})\right) \\
&\quad + \sum_{r<s} (-1)^{r+s} \psi^{(p)}\left(\mu(T_{A_r},\, T_{A_s}),\, T_{A_0}, \ldots, \hat{T}_{A_r}, \ldots, \hat{T}_{A_s}, \ldots, T_{A_p}\right)
\end{aligned}
$$

Examples: ($\phi \in C^1\,,\ \psi \in C^2$)

$$
\begin{aligned}
s_\mu \triangleright \phi(A_1, A_2) &= [A_1, \phi(A_2)] - [A_2, \phi(A_1)] - \phi([A_1, A_2]) \\
s_\mu \triangleright \psi(A_1, A_2, A_3) &= [A_1, \psi(A_2, A_3)] - [A_2, \psi(A_1, A_3)] + [A_3, \psi(A_1, A_2)] \\
&\quad - \psi([A_1, A_2], A_3) + \psi([A_1, A_3], A_2) - \psi([A_2, A_3], A_1)
\end{aligned}
$$

Jacobi for $\mu \Rightarrow \boxed{s_\mu^2 = 0}$

• Cohomology groups

Jacobi for $\mu_t \quad \Rightarrow \quad \boxed{s_{\mu_0} \triangleright \psi_1 = 0}$

$\Rightarrow \quad \psi_1 \in Z^2(V, s_\mu)$ (*2-cocycle* — similarly Z^p)

where $\mu_t(X,Y) \equiv [X,\, Y]_t = [X,\, Y]_0 + \psi_1(X,Y)t + \ldots$

The deformation is trivial iff $\exists \phi \in C^1(V)$ s.t. $\boxed{\psi_1 = s_{\mu_0} \triangleright \phi}$

$\Rightarrow \quad \psi_1 \in B^2(V, s_\mu)$ (*2-coboundary*, *trivial 2-cocycle* — similarly B^p)

Non-trivial deformations generated by non-trivial 2-cocycles

(tangent space interpretation, slide 6)

$H^p \equiv Z^p/B^p$ *p-th cohomology group of* $\mathcal{G}_0$

$H^2(\mathcal{G}_0)$ trivial $\quad \Rightarrow \quad \mathcal{G}_0$ stable (converse *not* true) $\quad \Rightarrow$

semisimple Lie algebras stable

• The $\barwedge$ product

$\barwedge\colon C^p \times C^q \longrightarrow C^{p+q-1}$

$$\alpha \barwedge \beta(X_0, \ldots, X_{m+n}) = \sum_\sigma \operatorname{sgn}(\sigma)\, \alpha\big(\beta(X_{\sigma(0)}, \ldots, X_{\sigma(n)}), X_{\sigma(n+1)}, \ldots, X_{\sigma(m+n)}\big)$$

Graded commutator:

$$[\![\alpha, \beta]\!] = \alpha \barwedge \beta - (-1)^{mn} \beta \barwedge \alpha \qquad (\alpha \in C^{m+1},\ \beta \in C^{n+1})$$

Jacobi for μ $\quad\Leftrightarrow\quad$ $\mu \barwedge \mu = \frac{1}{2}[\![\mu, \mu]\!] = 0$

In general: $s_\mu \triangleright \psi = (-1)^p [\![\mu, \psi]\!]$

Assume $[\![\mu, \mu]\!] = 0$ (μ Lie product). $\mu_t = \mu + \phi_t$ also Lie product iff

$$[\![\mu_t, \mu_t]\!] = 0 \qquad \Rightarrow \qquad \boxed{s_\mu \triangleright \phi_t - \frac{1}{2}[\![\phi_t, \phi_t]\!] = 0}$$ *deformation equation*

• Obstructions and H^3

$$\mu_t = \mu + \phi_t \,, \qquad\qquad \phi_t = \sum_{n=1}^{\infty} \phi_n t^n$$

Defomation equation $\quad \Rightarrow$

$$s_\mu \triangleright \phi_1 = 0$$
$$s_\mu \triangleright \phi_2 = \frac{1}{2} [\![\phi_1 , \, \phi_1]\!]$$
$$s_\mu \triangleright \phi_3 = [\![\phi_1 , \phi_2]\!]$$
$$\vdots$$

$H^3(\mathcal{G}) \neq 0 \quad \Rightarrow \quad \phi_1 \in H^2(\mathcal{G})$ might be *non-integrable*

Notice:

$$[\![\phi_1 , \, \phi_1]\!] = 0 \quad \Rightarrow \quad \mu + \phi_1 t \;\; \text{Lie product}$$

• Coboundary operator as exterior covariant derivative

Π^A: left invariant 1-forms, $\langle \Pi^A, T_B \rangle = \delta_B{}^A$

$$\psi^{(p)} \to \psi^B \otimes T_B \equiv \frac{1}{p!}\psi_{A_1 \ldots A_p}{}^B \, \Pi^{A_1} \ldots \Pi^{A_p} \otimes T_B$$

$$s_\mu \to \nabla = d + \Omega \,, \qquad \Omega^A{}_B = f_{RB}{}^A \Pi^R$$

(components defined by: $\psi^{(p)}(T_{A_1}, \ldots, T_{A_p}) = \psi_{A_1 \ldots A_p}{}^B T_B$)

Example: Galilean kinematics

$$[J_a, J_b] = i\,\epsilon_{ab}{}^c J_c \,, \qquad [J_a, K_b] = i\,\epsilon_{ab}{}^c K_c \,, \qquad [K_a, K_b] = 0$$

$$\mu = \frac{1}{2}\epsilon_{ab}{}^c \Pi^a \Pi^b \otimes J_c + \epsilon_{ab}{}^c \Pi^a \Pi^{\bar{b}} \otimes K_c$$

Only non-trivial 2-cocycle: $\chi_{\text{KKJ}} = \frac{1}{2}\epsilon_{ab}{}^c \Pi^{\bar{a}} \Pi^{\bar{b}} \otimes J_c$, with $[\![\chi_{\text{KKJ}}, \chi_{\text{KKJ}}]\!] = 0$

$$\Rightarrow \qquad [K_a, K_b]_t = i\, t \epsilon_{ab}{}^c J_c$$

Experiment says: $t = -\frac{1}{c^2}$

Heisenberg's Route

Classical relativity $\mathcal{G}_{\text{CR}}$ $(\hbar = 0)$

$$[J_{\mu\nu},\, J_{\rho\sigma}] = i\left(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}\right)$$
$$[J_{\rho\sigma},\, P_{\mu}] = i\left(g_{\mu\sigma}P_{\rho} - g_{\mu\rho}P_{\sigma}\right)$$
$$[J_{\rho\sigma},\, Z_{\mu}] = i\left(g_{\mu\sigma}Z_{\rho} - g_{\mu\rho}Z_{\sigma}\right),$$

plus M central.

Algorithm:

1. Most general 1-cochain: $\phi = \phi_A{}^B \Pi^A \otimes T_B$ (225 terms)
2. Most general 2-coboundary: $\psi = \nabla\phi$ (1008 terms)
3. Most general 2-cochain: $\chi = \chi_{AB}{}^C \Pi^A \Pi^B \otimes T_C$ (1575 terms)
4. Require χ a 2-cocycle, $\nabla\chi = 0$ (5672 equations in 1575 unknowns)
5. $\Rightarrow$ 221 2-cocycles χ_i. For each χ_i, solve $\chi_i = \psi$ (348075 equations)

6. $\Rightarrow$ only five χ_i non-trivial:

$$H^2(\mathcal{G}_{\mathrm{CR}}) = \{[0], [\psi_{\mathrm{H}}], [\psi_{\mathrm{PMZ}}], [\psi_{\mathrm{ZMP}}], [\psi_{\mathrm{PMP}}], [\psi_{\mathrm{ZMZ}}]\}$$

where

$$\begin{aligned}
\psi_{\mathrm{H}} &= \Pi^{\mu}\Pi_{\dot{\mu}} \otimes M \\
\psi_{\mathrm{PMZ}} &= \Pi^{\mu}\Pi^{M} \otimes Z_{\mu} \\
\psi_{\mathrm{ZMP}} &= \Pi^{\dot{\mu}}\Pi^{M} \otimes P_{\mu} \\
\psi_{\mathrm{PMP}} &= \Pi^{\mu}\Pi^{M} \otimes P_{\mu} \\
\psi_{\mathrm{ZMZ}} &= \Pi^{\dot{\mu}}\Pi^{M} \otimes Z_{\mu}
\end{aligned}$$

Deform along ψ_{H} only $\quad \rightarrow \quad \mathcal{G}_{\mathrm{PH}}(q)$

Stable Quantum Relativistic Kinematics

$\mathcal{G}_{\text{PH}}(q)$ (Poincaré plus Heisenberg):

$$[J_{\mu\nu},\, J_{\rho\sigma}] = i\left(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}\right)$$
$$[J_{\rho\sigma},\, P_{\mu}] = i\left(g_{\mu\sigma}P_{\rho} - g_{\mu\rho}P_{\sigma}\right)$$
$$[J_{\rho\sigma},\, Z_{\mu}] = i\left(g_{\mu\sigma}Z_{\rho} - g_{\mu\rho}Z_{\sigma}\right)$$
$$[P_{\mu}, Z_{\nu}] = i\, q\, g_{\mu\nu}M$$

$$\Rightarrow \quad \mu_{\text{PH}}(q) = \frac{1}{2}\Pi^{\alpha\rho}\Pi_{\rho}{}^{\beta}\otimes J_{\alpha\beta} + \Pi^{\alpha\rho}\Pi_{\rho}\otimes P_{\alpha} + \Pi^{\alpha\rho}\Pi_{\dot{\rho}}\otimes Z_{\alpha} + q\Pi^{\mu}\Pi_{\dot{\mu}}\otimes M$$

$$H^2(\mathcal{G}_{\text{PH}}(q)) = \{[0], [\zeta_1], [\zeta_2], [\zeta_3]\}$$

where

$$\zeta_1 = \Pi^{\mu}\Pi^{M}\otimes Z_{\mu} + \frac{q}{2}\Pi^{\mu}\Pi^{\nu}\otimes J_{\mu\nu}$$
$$\zeta_2 = -\Pi^{\dot{\mu}}\Pi^{M}\otimes P_{\mu} + \frac{q}{2}\Pi^{\dot{\mu}}\Pi^{\dot{\nu}}\otimes J_{\mu\nu}$$
$$\zeta_3 = \Pi^{\dot{\mu}}\Pi^{M}\otimes Z_{\mu} - \Pi^{\mu}\Pi^{M}\otimes P_{\mu} + q\Pi^{\mu}\Pi^{\dot{\nu}}\otimes J_{\mu\nu}$$

$[\![\zeta_i, \zeta_j]\!] = 0 \quad \Rightarrow \quad \mu(q, \vec{\alpha}) = \mu_{\text{PH}}(q) + \alpha_1\zeta_1 + \alpha_2\zeta_2 + \alpha_3\zeta_3$ Lie product.

Stable quantum relativistic kinematics:

$$
\begin{aligned}
[J_{\mu\nu},\, J_{\rho\sigma}] &= i\left(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}\right)\\
[J_{\rho\sigma},\, P_\mu] &= i\left(g_{\mu\sigma}P_\rho - g_{\mu\rho}P_\sigma\right)\\
[J_{\rho\sigma},\, Z_\mu] &= i\left(g_{\mu\sigma}Z_\rho - g_{\mu\rho}Z_\sigma\right)\\
[P_\mu, Z_\nu] &= i\,q g_{\mu\nu}M + i\,q\,\alpha_3 J_{\mu\nu}\\
[P_\mu, P_\nu] &= i\,q\,\alpha_1 J_{\mu\nu}\\
[Z_\mu, Z_\nu] &= i\,q\,\alpha_2 J_{\mu\nu}\\
[P_\mu, M] &= -i\,\alpha_3 P_\mu + i\,\alpha_1 Z_\mu\\
[Z_\mu, M] &= -i\,\alpha_2 P_\mu + i\,\alpha_3 Z_\mu
\end{aligned}
$$

provided $\alpha_3^2 \neq \alpha_1\alpha_2$.

When $\alpha_3^2 = \alpha_1\alpha_2$, $\chi = \zeta_1 + \zeta_2$ is a non-trivial integrable 2-cocycle

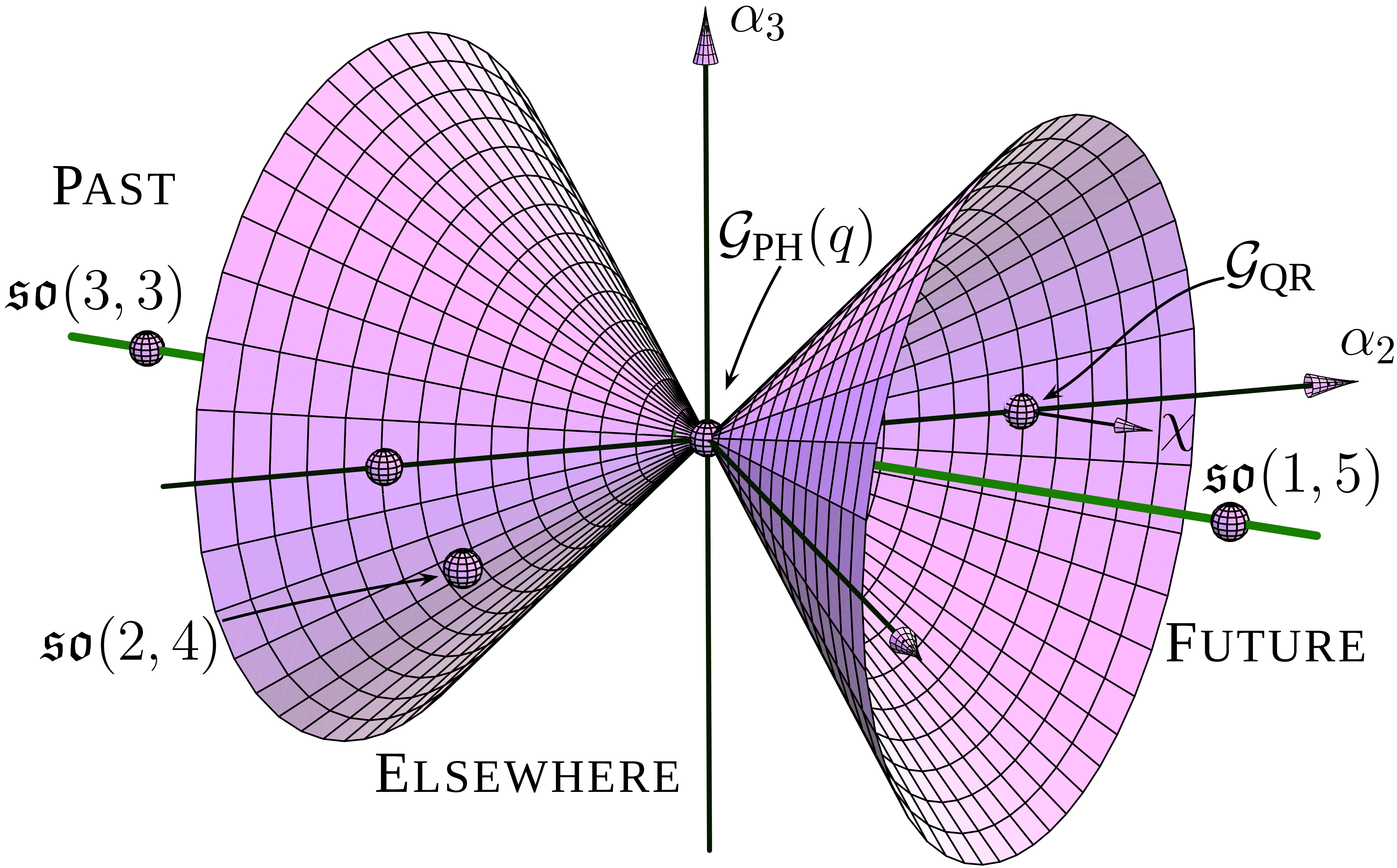

Figure 2: The $(\alpha_1, \alpha_2, \alpha_3)$-deformation space of $\mathcal{G}_{\mathrm{PH}}(q)$.

Physical Implications (in progress)

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- D. Ahluwalia-Khalilova, "A Freely Falling Frame at the Interface of the Gravitational and Quantum Realms", *Class. Quantum Grav.* **22** (2005) 1433-1450
- D. Ahluwalia-Khalilova, "Minimal Spatio-Temporal Extent of Events, Neutrinos, and the Cosmological Constant Problem", `hep-th/0505124` (honorable mention in the 2005 Essay Competition of the Gravity Research Foundation)

Standard wisdom ($q = 1$):

$$[P_\mu,\, P_\nu] = i\frac{1}{R^2}J_{\mu\nu} \qquad R = \frac{1}{\sqrt{\Lambda}}$$

$$[Z_\mu,\, Z_\nu] = i\ell_{\mathrm{P}}^2 J_{\mu\nu} \qquad \ell_{\mathrm{P}}^2 \equiv G$$

$\Rightarrow$ noncommutative spacetime, energy- momentum space

However:

$J_{\mu\nu}$, P_μ: primitive (extensive), *e.g.*, total angular momentum: $J_{\text{tot}} = J_1 + J_2$

Positions not primitive $\Rightarrow$ X_μ **_not Lie algebra generators_**

$$\text{Newtonian limit:} \quad X_{12} = \frac{M_1 X_1 + M_2 X_2}{M_1 + M_2}$$

$\Rightarrow \quad Z_\mu = X_\mu M$ primitive $\quad \left(M = \sqrt{P^\mu P_\mu}\,\right)$

$$[Z_\mu,\, Z_\nu] = iq(X_\mu P_\nu - X_\nu P_\mu) = iqL_{\mu\nu}$$

Spinless particles: $\alpha_2 = 1$ (commutative spacetime!)

To Do List

- $\mathfrak{so}(1,5)$-representations, Casimirs
- Wigner like particle description
- Relativistic Z_μ? Higher spin? Zero mass?
- Non-commutative spacetime?
- Invariant length $\Rightarrow$ momentum cutoff?
- Invariant length + Lorentz contraction = ?
- Supersymmetry?